

Adjoints of rationally induced composition operators

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Abstract

We give an elementary proof of a formula recently obtained by Hammond, Moorhouse, and Robbins for the adjoint of a rationally induced composition operator on the Hardy space H^2 [Christopher Hammond, Jennifer Moorhouse, and Marian E. Robbins, *Adjoints of composition operators with rational symbol*, J. Math. Anal. App., 341 (2008) 626–639]. We discuss some variants and implications of this formula, and use it to provide a sufficient condition for a rationally induced composition operator adjoint to be a compact perturbation of a weighted composition operator.

Key words:

Composition operator, adjoint, rational function

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Introduction

We study composition operators $C_\varphi : f \rightarrow f \circ \varphi$ acting on the Hardy space H^2 of the open unit disc \mathbb{U} of the complex plane. Our goal is to provide a simple derivation for an intriguing formula, established recently by Hammond, Moorhouse, and Robbins [12], that describes the adjoints of composition operators induced by rational selfmaps of \mathbb{U} .

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The study of composition operator adjoints was initiated more than twenty years ago by Carl Cowen [6], who showed that if φ is linear-fractional then C_φ^* , the adjoint of C_φ on H^2 , has the form $M_g C_\sigma M_h^*$, where M_g and M_h are the operators of multiplication by simple rational functions g and h , bounded on \mathbb{U} , and σ is a linear-fractional selfmap of \mathbb{U} —all these functions being easily obtainable from the coefficients of φ (see also [8, Theorem 9.2, page 322]).

Cowen's work showed, in particular, that one should expect the adjoint of a composition operator to be substantially more complicated than just another composition operator; in the linear-fractional case it is the adjoint of a multiplication operator followed by a *weighted composition operator* $M_g C_\sigma$. Complications notwithstanding, Cowen's formula has provided an essential tool in a number of investigations involving linear-fractionally induced composition operators, for example: subnormality and co-subnormality [6], computation of norms (see Hammond's fundamental paper [11], and subsequent work [1,2,9] based upon it), essential normality [3], self-commutator properties [4], C^* algebras [14,15], and singular value decompositions [5].

Useful as Cowen's work has been, the study of adjoint formulas for composition operators has only recently been pushed beyond the linear-fractional case. Although several different approaches have been tried (see [16] and the references described therein), explicit formulas have been hard to come by, even for rationally induced operators.

Recent work of Cowen and Gallardo [7] (see also the discussion of [7] in [12]) made the point that for general rational selfmaps of the disc the determination of C_φ^* must involve what they called “multiple-valued weighted composition operators.” For a graphic demonstration of how multiple-valuedness can arise, consider the simplest rational selfmap of \mathbb{U} that is not linear-fractional: $\varphi(z) = z^2$. Elementary calculations show that for each function $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ in H^2 ,

$$C_\varphi^* f(z) = \sum_{n=0}^{\infty} \hat{f}(2n)z^n = \frac{1}{2}[f(\sqrt{z}) + f(-\sqrt{z})] \quad (z \in \mathbb{U}).$$

Thus for this example we can regard C_φ^* as a formal average of “composition operators” induced by branches of the square root function, with the inducing maps having singularities along a branch cut which, even when subjected to composition with an arbitrary $f \in H^2$, retain enough cancellation to yield a result that is holomorphic on the whole unit disc.

Hammond, Moorhouse, and Robbins [12] recently corrected a formula in [7] for the adjoint of a composition operator on H^2 induced by a rational function φ . The method of both papers centers on Cauchy integral representations of the H^2 inner product $\langle f, C_\varphi^* g \rangle$, for polynomials f and g . A case-based application of Cauchy's Theorem followed by a change of variables results

in a representation of the inner product involving a generalization of the map σ that figured in Cowen's linear-fractional theorem, the difficulty being that beyond the linear-fractional case this generalization turns out to be *multiple-valued*, so that its branching behavior needs to be taken into account.

Our initial goal here is to obtain the Hammond-Moorhouse-Robbins (HMR) formula in a straightforward algebraic fashion, resulting in a unified derivation that is considerable shorter than that in [12]. After setting out the necessary background material in the next section, we devote §2 to our proof of the HMR formula, discussing some of its variants in §3. In §4 we address the question of when C_φ^* is a sum involving “legitimate” weighted composition operators, after which we apply our results to the question of which rationally induced composition operators have adjoint equivalent, modulo compacts, to a weighted composition operator.

1 Background material

For expositional completeness we collect here some standard background material that will be needed in what follows.

1.1 MAPPING RATIONALLY. We will need to think of rational functions, even self-maps of the unit disc, as mappings taking the extended complex plane (i.e., the Riemann sphere) $\hat{\mathbb{C}}$ into itself. Throughout this paper, φ denotes an element of $\text{Rat}(\mathbb{U})$, the collection of all rational functions of one complex variable that map \mathbb{U} into itself. Associated with φ is its “exterior” map $\varphi_e := \rho \circ \varphi \circ \rho$, where $\rho : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ denotes inversion in the unit circle ($\rho(z) := 1/\bar{z}$). Thus φ_e maps the exterior \mathbb{U}_e of the closed unit disc (viewed as a subset of $\hat{\mathbb{C}}$) into itself. On the other hand it is easy to check that $\varphi_e^{-1}(\mathbb{U}) \subset \mathbb{U}$, a fact that will be of utmost importance to us.

1.2 REGULAR AND CRITICAL VALUES. For any rational function R , expressed as a quotient of relatively prime polynomials, the *degree* of R is the larger of the degrees of its numerator and denominator. It is an elementary exercise to show that if the degree of R is d then for each point $w \in \hat{\mathbb{C}}$ the inverse image $R^{-1}(\{w\})$ has, counting multiplicities, exactly d points (see [10, Problems 25–32, pp. 181–182] for example). If $R^{-1}(\{w\})$ has d *distinct* points we will say w is a *regular value* of R . Clearly, for any rational function, all but finitely many points of $\hat{\mathbb{C}}$ are regular values. We call those values that are not regular *critical values* (perhaps more commonly these are called “branch points of R^{-1} ”).

By elementary function theory, $w \in \hat{\mathbb{C}}$ is a critical value of R iff $R^{-1}(\{w\})$ contains at least one point having no neighborhood on which R is univalent. Such a point is called a *critical point*. Thus a value is critical for R if and only if its preimage contains a critical point.

Suppose that w is a regular value of R , so that $R^{-1}(\{w\})$ consists of distinct points $\{z_1, z_2, \dots, z_d\}$ in $\hat{\mathbb{C}}$. Then, again by elementary function theory, each point z_j is the center of an open disc Δ_j to which R has a univalent restriction, hence the restriction of R to Δ_j has a holomorphic inverse σ_j on $R(\Delta_j)$; it is called a *branch* of R^{-1} defined on $R(\Delta_j)$. Upon taking the intersection of the R -images of the d discs Δ_j we obtain a common neighborhood of w on which all the distinct branches $\{\sigma_j\}_{j=1}^d$ are defined.

1.3 BRANCHING OUT. By way of introduction, suppose more generally that V and W are arbitrary sets and $f : V \rightarrow W$ is a function that is one-to-one on a subset V_1 of V . Then the restriction of f to V_1 has an inverse σ_1 mapping $\Omega := f(V_1)$ back onto V_1 . Let's call σ_1 a *branch of f^{-1} defined on Ω* . It's easy to check that if σ_2 is another branch of f^{-1} defined on Ω , with $V_2 = \sigma_2(\Omega)$, then $\sigma_1 \equiv \sigma_2$ on $f(V_1 \cap V_2)$.

Suppose, in particular that V is an open subset of $\hat{\mathbb{C}}$, f is holomorphic on V , and that σ_1 and σ_2 are branches of f^{-1} defined on a connected open subset Ω of $\hat{\mathbb{C}}$. Then σ_1 and σ_2 are both holomorphic on Ω , the sets V_1 and V_2 of the last paragraph being, respectively, $\sigma_1(\Omega)$ and $\sigma_2(\Omega)$. Thus if $V_1 \cap V_2$ is nonempty, then its image under f is, by the last paragraph, a nonvoid open subset of Ω on which $\sigma_1 \equiv \sigma_2$. By the uniqueness theorem for holomorphic functions, $\sigma_1 \equiv \sigma_2$ on all of Ω . In summary:

Proposition 1 *Suppose f is holomorphic on an open subset of $\hat{\mathbb{C}}$ and σ_1, σ_2 are two branches of f^{-1} defined on a connected open subset Ω of $\hat{\mathbb{C}}$. Then either $\sigma_1(\Omega) \cap \sigma_2(\Omega) = \emptyset$ or $\sigma_1 \equiv \sigma_2$ on Ω .*

1.4 CONTINUING BRANCHES. Suppose R is a rational function. Let $\text{reg}(R)$ denote the collection of points in $\hat{\mathbb{C}}$ that are regular values of R . The discussion of regular values in §1.2 shows that $\text{reg}(R)$ is an open subset of $\hat{\mathbb{C}}$; furthermore $\text{reg}(R)$ supports “arbitrary continuation” of branches of R^{-1} . More precisely:

Proposition 2 *Suppose R is a rational function and $\gamma : [0, 1] \rightarrow \text{reg}(R)$ a continuous curve. Suppose σ is a branch of R^{-1} defined on an open disc centered at $\gamma(0)$ and lying in $\text{reg}(R)$. Then σ has a holomorphic continuation along γ .*

Proof Forget the curve for a moment and just suppose we have two open discs Δ_1 and Δ_2 contained in $\text{reg}(R)$, each of which supports d distinct branches of R^{-1} ; say $\{\sigma_j^{(1)}\}_{j=1}^d$ on Δ_1 and $\{\sigma_j^{(2)}\}_{j=1}^d$ on Δ_2 . Suppose $\Delta_1 \cap \Delta_2 \neq \emptyset$.

Claim: For each index j the branch $\sigma_j^{(1)}$ of R^{-1} has a holomorphic extension to $\Delta_1 \cup \Delta_2$.

Proof of Claim. Fix $w_0 \in \Delta_1 \cap \Delta_2$. Then for $i = 1, 2$;

$$R^{-1}(\{w_0\}) = \{\sigma_j^{(i)}(w_0)\}_{j=1}^d,$$

where the set on the right has d distinct elements. Thus there is an index k such that $\sigma_j^{(1)}(w_0) = \sigma_k^{(2)}(w_0)$. Now both $\sigma_j^{(1)}$ and $\sigma_k^{(2)}$ are branches of R^{-1} defined on the connected open set $\Delta_1 \cap \Delta_2$, and by our choice of k , the images of that intersection under the two branches has nonvoid intersection. Thus by Proposition 1, $\sigma_j^{(1)} \equiv \sigma_k^{(2)}$ on the intersection, and so the function σ defined on $\Delta_1 \cup \Delta_2$ by:

$$\sigma = \begin{cases} \sigma_j^{(1)} & \text{on } \Delta_1 \\ \sigma_k^{(2)} & \text{on } \Delta_2 \end{cases}$$

gives the desired holomorphic extension of $\sigma_j^{(1)}$ to $\Delta_1 \cup \Delta_2$.

Now let's talk about the the curve γ . By its compactness in the open set $\text{reg}(R)$ and the discussion of §1.2, it can be covered by a chain of open discs $\{\Delta_j\}_{j=1}^n$ lying in $\text{reg}(R)$, each supporting d distinct branches of R^{-1} , with Δ_0 being the open disc on which the branch σ of R^{-1} is initially defined, and each disc having nontrivial intersection with its successor. Then successive application of the *Claim* effects a holomorphic continuation of σ along the curve γ . \square

Corollary 3 Suppose R is a rational function and σ is a branch of R^{-1} defined on some disc $\Delta \subset \text{reg}(R)$. Suppose W is a simply connected domain containing Δ and lying in $\text{reg}(R)$. Then σ has a holomorphic continuation to a branch of R^{-1} defined on W .

Proof Proposition 2, along with the Monodromy Theorem [13, Theorem 10.3.1, page 12], yields the desired continuation of σ to W . That this continuation is a branch of R^{-1} follows quickly from the uniqueness theorem for holomorphic functions (or, perhaps more ostentatiously, from the “law of permanence of functional equations” [13, §10.7, pp. 31–36]). \square

1.5 H^2 AND ITS REPRODUCING KERNELS. The space H^2 is the collection of functions holomorphic on \mathbb{U} with square-summable Maclaurin-coefficient sequence. The norm $\|f\|$ of a function $f \in H^2$ with Maclaurin series $f(z) =$

$\sum_{n=0}^{\infty} \hat{f}(n)z^n$ is defined by:

$$\|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 ;$$

this norm makes H^2 into a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \quad (f, g \in H^2).$$

A simple computation shows that for each $z \in \mathbb{U}$ and $f \in H^2$,

$$f(z) = \langle f, K_z \rangle \quad \text{where} \quad K_z(w) := \frac{1}{1 - \bar{z}w} \quad (w \in \mathbb{U}). \quad (1)$$

The function K_z (clearly in H^2) is called the *reproducing kernel* for the point z .

1.6 THE “KERNEL” OF OUR ARGUMENT. The importance to our work of reproducing kernels stems from the observation that

$$C_{\varphi}^* f(z) = \langle C_{\varphi}^* f, K_z \rangle = \langle f, C_{\varphi} K_z \rangle. \quad (2)$$

Now $C_{\varphi} K_z(w) = (1 - \bar{z}\varphi(w))^{-1}$ is a rational function of w whose partial fraction expansion we will determine explicitly. The desired formula for $C_{\varphi}^* f(z)$ will emerge upon substitution of this expansion into (2); the details are in the next section.

1.7 MUCH ADO ABOUT ZERO. Although the general formula for $C_{\varphi}^* f(z)$ can be quite complicated, for $z = 0$ it could not be simpler. Upon substituting $z = 0$ into (2) and noting that $K_0 \equiv 1$, we obtain

$$C_{\varphi}^* f(0) = f(0) \quad (f \in H^2),$$

an observation we will need several times in the sequel.

1.8 THE BACKWARD SHIFT. In §3 we will derive alternate formulae for $C_{\varphi}^* f$, one of which will involve the backward shift operator B , defined for $f \in H^2$ by:

$$(Bf)(z) := \begin{cases} \frac{f(z) - f(0)}{z} & \text{if } z \in \mathbb{U} \setminus \{0\} \\ f'(0) & \text{if } z = 0. \end{cases}$$

This operator is so named because it has the effect of shifting Maclaurin coefficient sequences “back one step,” i.e.,

$$B : \sum_{k=0}^{\infty} \hat{f}(k)z^k \rightarrow \sum_{k=0}^{\infty} \hat{f}(k+1)z^k \quad (f \in H^2).$$

It is easy to check that B is the H^2 -adjoint of the multiplication operator M_z defined, albeit with some abuse of notation, by:

$$M_z : f(z) \rightarrow zf(z) \quad (f \in H^2).$$

Because M_z has the effect of moving Maclaurin coefficient sequences “forward one step” it is called—not surprisingly—the “forward shift.”

2 Main Theorem

Our goal in this section is to give an elementary proof of the formula for C_{φ}^* proved by Hammond, Moorhouse, and Robbins [12].

2.1 THE EXTERIOR MAP. In order to state the result efficiently, recall from §1.1 that each rational selfmap φ of \mathbb{U} has a companion rational selfmap $\varphi_e = \rho \circ \varphi \circ \rho$ of \mathbb{U}_e , where ρ is the mapping of inversion in the unit circle. Note that the degree of φ_e is the same as that of φ , and that a point of $\hat{\mathbb{C}}$ is a regular value of φ_e if and only if its reflection in the unit circle is a regular value of φ .

2.2 THE MAIN THEOREM. Recall that if $z_0 \in \mathbb{U}$ is a regular value of φ_e and φ_e has degree d , then, as we saw in §1.2, there is a neighborhood V of z_0 , which we assume to be connected, on which are defined d distinct branches $\{\sigma_j\}_{j=1}^d$ of φ_e^{-1} . The univalence of these branches insures that their derivatives vanish nowhere on V . Note that Proposition 1 shows each element of V is a regular value of φ_e and that our earlier observation $\varphi_e^{-1}(\mathbb{U}) \subset \mathbb{U}$ shows $\sigma_j(z) \in \mathbb{U}$ for every $z \in V \cap \mathbb{U}$.

Theorem 4 ([12]) *Let $\varphi \in \text{Rat}(\mathbb{U})$ have degree $d > 0$ and let $f \in H^2$. Suppose $z_0 \in \mathbb{U}$ is a regular value of φ_e and $V \subset \mathbb{U}$ is any connected neighborhood of z_0 on which are defined d distinct branches $\{\sigma_j\}_{j=1}^d$ of φ_e^{-1} .*

(a) *If $\varphi(\infty) \neq \infty$, then for all $z \in V \setminus \{1/\overline{\varphi(\infty)}\}$*

$$C_{\varphi}^* f(z) = \frac{f(0)}{1 - \overline{\varphi(\infty)}z} + z \sum_{j=1}^d \frac{\sigma'_j(z)}{\sigma_j(z)} f(\sigma_j(z)) . \quad (3a)$$

(b) If $\varphi(\infty) = \infty$, then for all $z \in V$

$$C_\varphi^* f(z) = \begin{cases} z \sum_{j=1}^d \frac{\sigma'_j(z)}{\sigma_j(z)} f(\sigma_j(z)) & \text{if } z \neq 0 \\ f(0) & \text{if } z = 0 \end{cases} \quad (3b)$$

2.3 DISCUSSION. Before we begin a rigorous proof, some words need to be said about these formulas. Note first that they must reflect, as they do, the fact that $C_\varphi^* f(0) = f(0)$ for every $f \in H^2$ (see §1.7). In formula (3b), since $C_\varphi^* f$ is continuous at 0, the second line on the right must be a limiting case of the first one. This may be proved independently as follows.

Formula (3b) and continuity at the origin. The first line of (3b) appears as a natural way of interpreting (3a) when $\varphi(\infty) = \infty$. In this case $\varphi_e(0) = 0$ so, assuming that V contains 0, we conclude that exactly one of the σ_j 's has value zero at the origin (Proposition 1). For this j we see that $z\sigma'_j(z)/\sigma_j(z) \rightarrow \sigma'_j(0)/\sigma_j'(0) = 1$ as $z \rightarrow 0$ (recall that univalence of σ_j ensures $\sigma'_j(0) \neq 0$). The other such terms clearly converge to 0 as $z \rightarrow 0$, hence the first formula of (3b) does indeed tend to $f(0)$ as $z \rightarrow 0$.

Remarks on equation (3a). Since $\varphi_e(0) = 1/\overline{\varphi(\infty)}$, the hypothesis $\varphi(\infty) \neq \infty$ means that $\varphi_e(0) \neq 0$ while the hypothesis $z \neq 1/\overline{\varphi(\infty)}$ means that no denominator $\sigma_j(z)$ on the right-hand side of (3a) is zero. If, however, $\varphi_e(0)$ were to lie in V , and we were to allow z to take that value, then one (and exactly one, by Proposition 1) of the $\sigma_j(z)$'s would be zero. Since $\sigma'_j(z) \neq 0$ this would, since $z \neq 0$, introduce in the second term on the right-hand side of (3a) a pole of order 1 at z . But now $\varphi(\infty) = 1/\overline{z} \in \mathbb{U}_e$, which provides the first term on the right-hand side of (3a) with a pole of order 1 at z , and since the left-hand side of that equation is holomorphic on \mathbb{U} , these poles must cancel each other. Thus formula (3a) could be considered valid at $z = 1/\overline{\varphi(\infty)}$ in the sense that the left-hand side is the limit of the right as z approaches $1/\overline{\varphi(\infty)}$.

A convenient choice of V . Let \tilde{V} be the unit disk with radial slits from each critical value of φ_e to the unit circle removed. Then \tilde{V} is a simply connected domain and Corollary 3 shows that φ_e^{-1} has d distinct branches $\{\sigma_j\}_{j=1}^d$ defined on \tilde{V} . We can assert that formula (3a) is valid for all $z \in \tilde{V}$ provided we interpret the formula according to the discussion provided above. Note that if φ_e has no critical values in \mathbb{U} , then $\tilde{V} = \mathbb{U}$ and the branches $\{\sigma_j\}_{j=1}^d$ of φ_e^{-1} become holomorphic self-maps of \mathbb{U} .

2.4 PROOF OF MAIN THEOREM. Let $z_0 \in \mathbb{U}$ be a regular value of φ_e and $V \subset \mathbb{U}$ be any connected neighborhood of z_0 on which are defined d distinct branches $\{\sigma_j\}_{j=1}^d$ of φ_e^{-1} . Let $z \in V$ so that z is a regular value of φ_e whose d preimages under φ_e are, of course, $\{\sigma_j(z)\}_{j=1}^d$. Given the discussion of §2.3, we may assume that $z \neq 0$ and we shall also assume that $z \neq 1/\overline{\varphi(\infty)}$, so that $C_\varphi K_z(w)$ has a finite limit as $w \rightarrow \infty$.

Since $z \neq 0$ it is easy to check that $C_\varphi K_z(w)$, viewed as a rational function of w , has the same degree as φ (and therefore of φ_e); it has its poles at the points w for which $\varphi(w) = 1/\overline{z} := \rho(z)$, i.e. at the points of $\varphi^{-1}(\{\rho(z)\})$. Note these poles are simple since $1/\overline{z}$ is a regular value of φ . As we pointed out in §1.6, the crucial step of our proof will be to determine the partial fraction expansion of $C_\varphi K_z$.

Because $z \in \mathbb{U}$ is a non-zero regular value of φ_e , its reflection $\rho(z) \in \mathbb{U}_e$ is a finite regular value of φ , with $\varphi^{-1}(\rho(z))$ consisting of the d distinct points in $\{1/\overline{\sigma_j(z)}\}_{j=1}^d$, all of which are finite by our assumption that $z \neq 1/\overline{\varphi(\infty)}$. For notational convenience, we set

$$1/\overline{\sigma_j(z)} = w_j$$

for each $j \in \{1, 2, \dots, d\}$.

Thus we have this partial fraction expansion for $C_\varphi K_z(w) = 1/(1 - \overline{z}\varphi(w))$:

$$C_\varphi K_z(w) = \alpha + \sum_{j=1}^d \frac{\beta_j}{w - w_j} = \alpha + \sum_{j=1}^d \frac{\beta_j/w_j}{w/w_j - 1}, \quad (4)$$

where

$$\alpha = K_z(\varphi(\infty)) = \frac{1}{1 - \overline{z}\varphi(\infty)} \in \mathbb{C}, \quad (5)$$

with the β_j 's, which also depend on z , to be determined shortly. Note for further reference that $\alpha = 0$ in case $\varphi(\infty) = \infty$.

Since each of the points $\rho(w_j)$ is in \mathbb{U} , equation (4) can be rewritten in terms of the reproducing kernels for these points:

$$C_\varphi K_z(w) = \alpha - \sum_{j=1}^d \frac{\beta_j}{w_j} K_{\rho(w_j)}(w). \quad (6)$$

Pending determination of the coefficients β_j , the following calculation will produce the formula for C_φ^* . First, substitute (6) into (2) to obtain, for $f \in H^2$,

$$C_\varphi^* f(z) = \langle f, C_\varphi K_z \rangle = \langle f, \alpha - \sum_{j=1}^d \frac{\beta_j}{w_j} K_{\rho(w_j)} \rangle.$$

On the right-hand side of this equation interchange the sum and inner product, and use the reproducing kernel formula (1) to evaluate the resulting inner products. The result, in view of (5), is:

$$C_\varphi^* f(z) = \frac{f(0)}{1 - \overline{\varphi(\infty)}z} - \sum_{j=1}^d \overline{\beta_j} \rho(w_j) f(\rho(w_j)). \quad (7)$$

Evaluation of the β_j 's. Because we are assuming that the poles $\{w_j\}_{j=1}^d$ of $K_z \circ \varphi$ are all simple, this is an easy residue calculation. From (4) it's just:

$$\beta_j = \lim_{w \rightarrow w_j} \frac{(w - w_j)}{1 - \overline{z}\varphi(w)} = -\frac{1}{\overline{z}\varphi'(w_j)}, \quad (8)$$

where the last equality comes from the fact that $1 - \overline{z}\varphi(w_j) = 0$.

Here is the result produced by (7) and (8):

$$C_\varphi^* f(z) = \frac{f(0)}{1 - \overline{\varphi(\infty)}z} + \frac{1}{z} \sum_{j=1}^d \overline{(w_j \varphi'(w_j))^{-1}} f(1/\overline{w_j}), \quad (9)$$

where $\{w_j\}_{j=1}^d = \{1/\overline{\sigma_j(z)}\}_{j=1}^d = \varphi^{-1}(\{1/\overline{z}\})$.

Equation (9) easily produces the promised formula for $C_\varphi^* f(z)$. We require the auxiliary map

$$\overline{\varphi}(w) := \overline{\varphi(\bar{w})}$$

as well as the observation that

$$\overline{\varphi}'(w) = \overline{\varphi'(\bar{w})}. \quad (10)$$

For each $w \in V$ and $j \in \{1, 2, \dots, d\}$, we have $\varphi_e(\sigma_j(w)) = w$, which may be rewritten

$$\overline{\varphi}(1/\sigma_j(w)) = 1/w \text{ for } w \in V \setminus \{0\}. \quad (11)$$

Differentiating both sides of (11), using (10), and substituting z for w , we obtain for each j ,

$$\overline{\varphi'(w_j)} = \frac{\sigma_j(z)^2}{\sigma'_j(z)z^2},$$

which, together with $\sigma_j(z) = 1/\overline{w_j}$ and (9), yields formulas (3a) and (3b).

Note that our proof does not need to treat the case $\varphi(\infty) = \infty$ separately, since this is just the case $\alpha = 0$ of (4). It only remains to note that in this case the value $z = 1/\overline{\varphi(\infty)} = 0$ is now allowed, this issue having been addressed in §2.3 \square

We can produce a more compact formula for C_φ^* by applying Theorem 4 and the observation that whenever $z \in \mathbb{U}$ is a regular value of φ_e and σ_j is a branch of φ_e^{-1} defined near z , then $\varphi'_e \circ \sigma_j = 1/\sigma'_j$. Here is the the result.

Corollary 5 *Let $\varphi \in \text{Rat}(\mathbb{U})$ have degree $d > 0$. Suppose $z \in \mathbb{U}$ is a regular value of φ_e and $f \in H^2$. Then:*

(a) *If $\varphi(\infty) \neq \infty$ and $z \neq 1/\overline{\varphi(\infty)}$, then*

$$C_\varphi^* f(z) = \frac{f(0)}{1 - \overline{\varphi(\infty)} z} + z \sum_{w \in \varphi_e^{-1}(\{z\})} \frac{f(w)}{w \varphi'_e(w)}. \quad (12a)$$

(b) *If $\varphi(\infty) = \infty$ then*

$$C_\varphi^* f(z) = \begin{cases} z \sum_{w \in \varphi_e^{-1}(\{z\})} \frac{f(w)}{w \varphi'_e(w)} & \text{if } z \neq 0 \\ f(0) & \text{if } z = 0 \end{cases} \quad (12b)$$

2.5 BLASCHKE PRODUCTS. Equations (12) have historical precedent in work of McDonald [17], who considered the special case of φ a finite Blaschke product, i.e., a rational function of the form

$$\varphi(z) = \omega \prod_{j=1}^d \frac{a_j - z}{1 - \overline{a_j} z}, \quad (13)$$

where $\omega \in \partial\mathbb{U}$ and $a_j \in \mathbb{U}$ for $1 \leq j \leq d$. In this case $\varphi_e = \varphi$ (since $\varphi(\partial\mathbb{U}) = \partial\mathbb{U}$), whereupon Corollary 5 yields formulae that are close in spirit to McDonald's result, but with right-hand sides that are now more explicitly computable.

Example 6 Following McDonald [17], we consider Blaschke products of degree 2 with $\varphi(0) = 0$, i.e., with $d = 2$ and $a_1 = 0$ in (13). To simplify matters further let's take, in that equation, $\omega = 1$ and $a_2 = 1/2$, so our map becomes

$$\varphi(z) = z \frac{1 - 2z}{2 - z}.$$

Since $\varphi(\infty) = \infty$ we will compute C_φ^* using equation (3b), which we write formally as:

$$C_\varphi^* = M_{g_1} C_{\sigma_1} + M_{g_2} C_{\sigma_2} \quad (14)$$

where $g_j(z) = z \sigma'_j(z)/\sigma_j(z)$, and now, because $\varphi_e = \varphi$, the σ_j 's are branches

of φ^{-1} . We find

$$\sigma_1(z) = \frac{1+z+\sqrt{\Delta(z)}}{4} \quad \text{and} \quad \sigma_2(z) = \frac{1+z-\sqrt{\Delta(z)}}{4},$$

where $\Delta(z) = z^2 - 14z + 1$. If we take the square root function in these formulas to have its branch cut along the positive real axis, then $z \mapsto \sqrt{\Delta(z)}$ is holomorphic on the complex plane cut by the rays $(-\infty, 7 - 4\sqrt{3}]$ and $[7 + 4\sqrt{3}, \infty)$, and is discontinuous at each point of the cut. In particular, each of the σ_j 's is holomorphic on the slit disc $\mathbb{U} \setminus (-1, 7 - 4\sqrt{3}]$, and discontinuous at each point of the removed interval. So for this particular example, none of the elements on the right-hand side of (14) has anything more than formal meaning. As we will see in the next section, this happens for *every* finite Blaschke product of degree ≥ 2 . For the moment let's just notice that $7 + 4\sqrt{3}$ is a critical value of $\varphi = \varphi_e$ that lies in \mathbb{U}_e .

3 Variants

In this section we discuss some variants of the Hammond-Moorhouse-Robbins formulae (3a)–(3b). In case $\varphi(\infty) \in \mathbb{U}_e$ and $\varphi(\infty) \neq \infty$, there is, as we discussed in §2.3, the phenomenon of pole cancellation occurring on the right-hand side of (3a).

3.1 FINESSING THE “POLE PROBLEM.” We focus, for the moment, on the subspace H_0^2 consisting of functions $f \in H^2$ with $f(0) = 0$. For $f \in H_0^2$ the first term on the right-hand side of (3a) disappears, so with Ω_φ defined, for all $z \in \mathbb{U} \setminus \{1/\overline{\varphi(\infty)}\}$ that are regular values of φ_e , by

$$\Omega_\varphi f(z) := \sum_{j=1}^d \frac{\sigma'_j(z)}{\sigma_j(z)} f(\sigma_j(z)) = \sum_{w \in \varphi_e^{-1}(\{z\})} \frac{f(w)}{w \varphi_e'(w)} \quad (15a)$$

we can write $C_\varphi^* f(z) = z \Omega_\varphi f(z)$ (for such z). We know from §1.7 that $C_\varphi^* f(0) = f(0) = 0$, so the function $z \rightarrow C_\varphi^* f(z)/z$ extends $\Omega_\varphi f$ holomorphically to all of \mathbb{U} . *We will continue to use the notation $\Omega_\varphi f$ to denote this extension.*

Of particular interest, for $f \in H_0^2$, is the value of our newly extended $\Omega_\varphi f$ at $z_0 := 1/\overline{\varphi(\infty)} = \varphi_e(0)$ when that troublesome point belongs to \mathbb{U} and is a regular value of φ_e . This situation has been discussed in the third paragraph of §2.3, where it was observed that $\sigma_j(z_0) = 0$ for *exactly one* index j which, without loss of generality, we now take to have the value 1. Then since $f(0) = 0$

we have

$$\lim_{z \rightarrow z_0} \frac{f(\sigma_1(z))}{\sigma_1(z)} = f'(0),$$

from which the first equation of (15a), when extended to $z = z_0$, becomes:

$$\Omega_\varphi f(z_0) = \sigma'_1(z_0) f'(0) + \sum_{j=2}^d \frac{\sigma'_j(z_0)}{\sigma_j(z_0)} f(\sigma_j(z_0)) \quad (15b)$$

with a similar extension, which we leave to the reader for the second equation of (15a).

The next result summarizes the discussion of the last two paragraphs, using the *backward shift* operator introduced in §1.8 to fold formulae (15a) and (15b) into a single expression.

Proposition 7 *If $\varphi \in \text{Rat}(\mathbb{U})$ then:*

- (a) Ω_φ is a bounded linear operator from H_0^2 into H^2 .
- (b) For each $f \in H_0^2$ and each $z \in \mathbb{U}$ that is a regular value of φ_e ,

$$\Omega_\varphi f(z) = \sum_{j=1}^d \sigma'_j(z) (Bf)(\sigma_j(z)).$$

Using Proposition 7 we obtain an alternative formula for C_φ^* in which both terms on the right-hand side represent bounded operators on H^2 .

Corollary 8 *If $\varphi \in \text{Rat}(\mathbb{U})$ then for each $f \in H^2$:*

$$C_\varphi^* f(z) = \frac{f(0)}{1 - \overline{\varphi(0)} z} + z \Omega_\varphi [f - f(0)](z) \quad (z \in \mathbb{U}) \quad (16a)$$

$$= \frac{f(0)}{1 - \overline{\varphi(0)} z} + z \sum_{j=1}^d \sigma'_j(z) (Bf)(\sigma_j(z)) \quad (z \in \text{reg}(\varphi_e) \cap \mathbb{U}) \quad (16b)$$

the functions $\{\sigma_j\}_{j=1}^d$ in (16b) being the d distinct branches of φ_e^{-1} defined (thanks to the φ_e -regularity of the value z) in a neighborhood of z .

Proof Upon replacing f in (2) by the constant function 1 we obtain:

$$C_\varphi^* 1(z) = \langle 1, K_z \circ \varphi \rangle = \overline{K_z(\varphi(0))} = \frac{1}{1 - \overline{\varphi(0)} z} \quad (z \in \mathbb{U}). \quad (17)$$

Then (16a) follows upon writing $f \in H^2$ as $f = f(0) + [f - f(0)]$, using the linearity of the operator C_φ^* , (17), and the definition of Ω_φ . Formula (16b) follows from (16a) by Proposition 7(b). \square

3.2 AN AMUSING IDENTITY. Upon comparing equations (3), or equivalently (12), with (16b) we find that for each regular value of φ_e lying in $\mathbb{U} \setminus \{1/\varphi(\infty)\}$

$$\frac{1}{1 - \overline{\varphi(0)} z} - \frac{1}{1 - \overline{\varphi(\infty)} z} = z \sum_{j=1}^d \frac{\sigma'_j(z)}{\sigma_j(z)} = z \sum_{w \in \varphi_e^{-1}(\{z\})} \frac{1}{w \varphi'_e(w)}.$$

This identity can be proved directly by, for example, observing that the contour integral

$$I(r) := \frac{1}{2\pi i} \int_{|w|=r} \frac{dw}{w(1 - z/\varphi_e(w))}$$

has the same value for each $r > 1$, and evaluating it: (i) by residues, and (ii) by letting $r \rightarrow \infty$.

3.3 WISHFUL THINKING. Proceeding in the the spirit of (14), we can regard formula (16b) to be a formal representation of C_φ^* as (modulo a rank-one operator) a sum of “operators” of the form $M_{z\sigma'_j} C_{\sigma_j} B$, each summand being reminiscent of Cowen’s original formula for the linear-fractional case. More precisely, in equation (16b) define the (legitimate) rank-one operator Λ_0 on H^2 by:

$$(\Lambda_0 f)(z) := \frac{f(0)}{1 - \overline{\varphi(0)} z} \quad (f \in H^2, z \in \mathbb{U}). \quad (18a)$$

Then (16b) can be rewritten, at least formally, as

$$C_\varphi^* = \Lambda_0 + \sum_{j=1}^d M_{h_j} C_{\sigma_j} B, \quad (18b)$$

where

$$h_j(z) := z\sigma'_j(z) \quad (z \in \mathbb{U}), \quad (18c)$$

and we make no immediate claims about the “global legitimacy” of the multiplication and composition operators that occur in the summation on the right-hand side of the equation.

In similar fashion, if we define

$$(\Lambda_\infty f)(z) := \frac{f(0)}{1 - \overline{\varphi(\infty)} z} \quad (19a)$$

then equation (3a) can be rewritten formally as:

$$C_\varphi^* = \Lambda_\infty + \sum_{j=1}^d M_{g_j} C_{\sigma_j}, \quad (19b)$$

where

$$g_j(z) := \frac{z\sigma'_j(z)}{\sigma_j(z)} , \quad (19c)$$

and now the legitimacy of even Λ_∞ as an operator on H^2 is an issue.

Of course our “wishful thinking” formulae, when applied to any $f \in H^2$, are valid for all but finitely many $z \in \mathbb{U}$ —that has been the point of our work up to now. In the next section we address the question of the “global legitimacy” of the formulae in the sense of operators on H^2 .

4 Legitimate operator equations

Let us agree to call equation (18b) a *legitimate operator equation* provided that all the multiplication and composition operators that occur on the right-hand side are bounded operators on H^2 , i.e., provided that the branches σ_j of φ_e^{-1} are all holomorphic on \mathbb{U} (in which case they are automatically *selfmaps* of \mathbb{U}), and the functions h_j are both holomorphic and bounded on \mathbb{U} . We make a similar definition for equation (19b), where now we must insist that $\varphi(\infty) \in \mathbb{U}$ in order to make Λ_∞ a bounded operator on H^2 . We saw in the Introduction (with the example $\varphi(z) = z^2$) and in Example 6 above that our “wishful thinking” formulae (18b) and (19b) need not always be legitimate operator equations. Here are two further examples, illustrating what can go right and what else can go wrong when we examine the legitimacy of our formulas for C_φ^* .

Example 9 Let $\varphi(z) = 1/(3 - z - z^2)$, so that

$$\varphi_e(w) := \rho(\varphi(\rho(w))) = 3 - \frac{1}{w} - \frac{1}{w^2} .$$

Upon solving the equation $z = \varphi_e(w)$ for w we find the two right-inverses of φ_e^{-1} in the form

$$\sigma_1(z) = \frac{1 + \sqrt{13 - 4z}}{2(3 - z)} \quad \text{and} \quad \sigma_2(z) = \frac{1 - \sqrt{13 - 4z}}{2(3 - z)} ,$$

where “ $\sqrt{-}$ ” denotes the principal branch of the square root function. Thus σ_1 and σ_2 are both defined and holomorphic on the slit plane $\mathbb{C} \setminus [\frac{13}{4}, \infty)$ except for an isolated singularity at $z = 3$, which is removable for σ_2 and is a simple pole for σ_1 . What’s important for our purposes is that σ_1 and σ_2 are nonzero and holomorphic on the closed unit disc. Because they are necessarily self-maps of \mathbb{U} , they induce composition operators on H^2 . Since $\varphi(\infty) = 0$ we see from Theorem 4 that for each $f \in H^2$ and $z \in \mathbb{U} \cap \text{reg}(\varphi_e)$:

$$C_\varphi^* f(z) = f(0) + g_1(z)f(\sigma_1(z)) + g_2(z)f(\sigma_2(z)) , \quad (20)$$

where $g_j(z) = z\sigma'_j(z)/\sigma_j(z)$. Since the σ 's are holomorphic and nonzero in a neighborhood of the closed unit disc, the g 's are bounded and holomorphic in \mathbb{U} . Because both sides of (20) represent holomorphic functions on \mathbb{U} , equality holds in (20) for all $z \in \mathbb{U}$; hence,

$$C_\varphi^* = \Lambda_\infty + M_{g_1}C_{\sigma_1} + M_{g_2}C_{\sigma_2} \quad (21)$$

where, because $\varphi(\infty) = 0$, Λ_∞ is the rank one operator that takes $f \in H^2$ to its value at the origin. So here, by contrast with the previous example, all the operators on the right-hand side of (21) are defined and bounded on H^2 , i.e., (21) is a legitimate operator equation.

Observe, for future reference, that in this example the critical points of φ are ∞ and $-1/2$, hence the critical values are $0 = \varphi(\infty)$ and $4/13 = \varphi(-1/2)$, both of which lie in \mathbb{U} . \square

The next example illustrates a situation intermediate between the behavior of Example 9 (“good”) and that of the Blaschke product in Example 6 (“bad”). In this one the critical values of φ lie in \mathbb{U} , with the result that the branches σ_1 and σ_2 of φ_e^{-1} are holomorphic on \mathbb{U} , but unfortunately $\varphi(\infty) \in \partial\mathbb{U}$, which we will see causes mischief elsewhere.

Example 10 Let $\varphi(z) = z^2/(3 - z - z^2)$. One quickly finds branches of φ_e in the form

$$\sigma_1(z) = \frac{1 + \sqrt{\Delta(z)}}{6} \quad \text{and} \quad \sigma_2(z) = \frac{1 - \sqrt{\Delta(z)}}{6},$$

where $\Delta(z) = 13 + 12z$ and, as before, we can take “ $\sqrt{}$ ” to denote the principal value of the square root. Thus both σ_1 and σ_2 are holomorphic on the slit plane $\mathbb{C} \setminus (-\infty, -\frac{13}{12}]$, which again contains the closed unit disc. But now Theorem 4 tells us that

$$C_\varphi^* f(z) = \frac{f(0)}{1+z} + g_1(z)f(\sigma_1(z)) + g_2(z)f(\sigma_2(z)), \quad (22a)$$

so even though the composition operators C_{σ_j} are both defined for $j = 1, 2$, at least one of the “weights” g_j must have a pole at -1 to balance, when $f(0) \neq 0$, the one provided by the first term on the right. One calculates that

$$g_1(z) = \frac{6z}{\sqrt{\Delta(z)}(1 + \sqrt{\Delta(z)})} \quad \text{and} \quad g_2(z) = \frac{-6z}{\sqrt{\Delta(z)}(1 - \sqrt{\Delta(z)})},$$

so that g_1 is bounded on \mathbb{U} , while g_2 has required the pole at -1 . Thus in the “wishful thinking” formula

$$C_\varphi^* = \Lambda_\infty f + M_{g_1}C_{\sigma_1} + M_{g_2}C_{\sigma_2} \quad (22b)$$

derived from (22a), all the operators make sense when acting on the space $H(\mathbb{U})$ of all functions holomorphic on \mathbb{U} , but on H^2 , even though M_{g_1} and

both composition operators act boundedly, M_{g_2} and Λ_∞ do not. In short, we may regard (22b) as a “legitimate operator equation” on $H(\mathbb{U})$, but not on H^2 .

On the other hand, note that $h_j(z) = z\sigma'_j(z)$ is bounded and analytic on \mathbb{U} for $j = 1$ and $j = 2$. Thus, we can regard (18b) as a legitimate operator equation for φ , σ_1 , and σ_2 as above. \square

The examples given so far show that the location both of critical values of φ and of $\varphi(\infty)$ are crucial to the way in which formulas (18b) and (19b) can be interpreted. Let us now make this official.

Definitions 11 (a) *We call a map $\varphi \in \text{Rat}(\mathbb{U})$ outer regular whenever its critical values all lie in \mathbb{U} , i.e., whenever each point of $\overline{\mathbb{U}_e} := \{|z| \geq 1\}$ is a regular value of φ .*

(b) *If, in addition, $\varphi(\infty) \in \mathbb{U}$ we say that φ is strongly outer regular.*

Thus the mapping of Example 9 is strongly outer regular, while that of Example 10 is outer regular, but, because $\varphi(\infty)$ lies on $\partial\mathbb{U}$, not strongly so. The Blaschke product of Example 6 has a critical value in \mathbb{U}_e , so it is not outer regular. This, in fact, is a feature of *every* finite Blaschke product of degree ≥ 2 .

Proposition 12 *Finite Blaschke products of degree ≥ 2 are never outer regular.*

Proof A finite Blaschke product φ of order > 1 must have a critical point somewhere in the finite plane (the derivative must vanish somewhere). This cannot occur on $\partial\mathbb{U}$, else φ could not take the unit disc into itself. Because the unit circle is taken onto itself, the reflection in the unit circle of any critical point is again a critical point, so there is a critical point in \mathbb{U}_e . Since Blaschke products are also selfmaps of \mathbb{U}_e there is a critical value in \mathbb{U}_e . \square

Here is our main result on outer regularity.

Theorem 13 *Suppose $\varphi \in \text{Rat}(\mathbb{U})$ has degree d .*

(a) *If φ is outer regular then (18b) is a legitimate operator equation.*

(b) *If, in addition, φ is strongly outer regular, then (19b) is a legitimate operator equation.*

Proof (a) The hypothesis of outer regularity on φ means that $r\mathbb{U}_e \subset \text{reg}(\varphi)$ for some $0 < r < 1$, hence $(1/r)\mathbb{U} \subset \text{reg}(\varphi_e)$. Thus Corollary 3 guarantees that φ_e^{-1} has d distinct branches $\{\sigma_j\}_{j=1}^d$ on $(1/r)\mathbb{U}$, each of which, as we noted in §1.1, maps \mathbb{U} into itself. Since each of these branches is holomorphic in a

neighborhood of $\overline{\mathbb{U}}$, each has a derivative bounded on \mathbb{U} , hence the functions h_j of formula (18b) are bounded on \mathbb{U} , and induce bounded multiplication operators on H^2 .

(b) The boundedness of Λ_∞ on H^2 is a consequence of the fact $\varphi(\infty) \in \mathbb{U}$ (*strong outer regularity*). The boundedness of the functions g_j will follow from that of the derivatives of the σ_j 's once we establish that each σ_j is bounded away from zero on \mathbb{U} . Now strong outer regularity guarantees that $\varphi_e(0) \in \mathbb{U}_e$. Fix j and suppose, for the sake of contradiction that $\sigma_j(z) = 0$ for some $z \in \overline{\mathbb{U}}$. Then, recalling that σ_j is a branch of φ_e^{-1} on a neighborhood of the *closure* of the unit disc, $z = \varphi_e(\sigma_j(z)) = \varphi_e(0) \in \mathbb{U}_e$, a contradiction. Thus σ_j does not vanish at any point of $\overline{\mathbb{U}}$, and so it is bounded away from zero on \mathbb{U} . \square

5 Adjoints modulo compacts

In this section we set out conditions under which adjoint composition operators are compact perturbations of simpler ones, e.g., weighted composition operators. The following example, which occurred in §4 as the poster child for the concept of strong outer regularity, motivates our work.

Example 14 *If $\varphi(z) = 1/(3 - z - z^2)$ then C_φ^* is a compact perturbation of a weighted composition operator.*

Proof Recall from Example 9 the branches σ_1 and σ_2 of φ_e^{-1} , both of which are holomorphic on a neighborhood of the closed unit disc. Note that $\sigma_1(1) = 1$, but that σ_2 takes the closed unit disc into \mathbb{U} . Thus C_{σ_2} is compact on H^2 [18, §2.2, page 23] while C_{σ_1} is not [18, §4.1, “Proposition” on page 56], so it follows from (21) that C_φ^* is a compact perturbation of the weighted composition operator $M_{g_1}C_{\sigma_1}$. \square

Here is the general result motivated by this example.

Corollary 15 *Suppose $\varphi \in \text{Rat}(\mathbb{U})$ is outer regular and maps exactly one point of $\partial\mathbb{U}$ to $\partial\mathbb{U}$. Then:*

- (a) *C_φ^* is a compact perturbation of an operator of the form $M_hC_\sigma B$ where σ is a holomorphic selfmap of \mathbb{U} that is a branch of φ_e^{-1} , the function $h : z \rightarrow z \rightarrow z\sigma'(z)$ is holomorphic and bounded on \mathbb{U} , and B is the backward shift on H^2 .*
- (b) *If, in addition, φ is strongly outer regular, then C_φ^* is a compact perturbation of the weighted composition operator M_gC_σ with σ as in part (a) and $g : z \rightarrow z\sigma'(z)/\sigma(z)$ a bounded holomorphic function on \mathbb{U} .*

Proof (a) The hypothesis on the boundary behavior of φ insures that $\varphi_e^{-1}(\partial\mathbb{U})$ contains exactly one point of $\partial\mathbb{U}$. By §1.1 and Proposition 1 the branches $\{\sigma_j\}_{j=1}^d$ map the closed unit disc $\overline{\mathbb{U}}$ to pairwise disjoint closed subsets of $\overline{\mathbb{U}}$, so the boundary behavior imposed on φ_e guarantees that exactly one of the maps σ_j , say σ_1 , takes a point of $\partial\mathbb{U}$ to $\partial\mathbb{U}$, with the other σ_j 's taking $\partial\mathbb{U}$ into \mathbb{U} . Thus, as in the discussion of Example 14, the composition operators C_{σ_j} are compact for $2 \leq j \leq d$, while C_{σ_1} is *not* compact. It follows that each of the operators $M_{z\sigma_j}C_{\sigma_j}B$ is compact ($2 \leq j \leq d$) so, upon noting that the rank-one operator Λ_0 is also compact, part (a) of Theorem 13 provides the desired representation for C_{φ}^* , with $\sigma := \sigma_1$.

(b) Use part (b) of Theorem 13 exactly as above, noting that, again as in Example 14 the strong outer regularity assumed for φ guarantees the boundedness, hence compactness, of the rank-one operator Λ_{∞} . \square

As an application of Corollary 15, consider the following generalization of Example 14.

Example 16 For $d = 2, 3, \dots$ let

$$\varphi(z) = \frac{1}{(d+1) - z - z^2 - \dots - z^d}.$$

Then C_{φ}^* is a compact perturbation of a weighted composition operator.

Proof One checks easily that φ maps the closed unit disc into $\mathbb{U} \cup \{1\}$, taking 1 to 1 and the rest of the unit circle into \mathbb{U} . Thus, in view of part (b) of Corollary 15, we need only show that φ is strongly outer regular. Since $\varphi(\infty) = 0 \in \mathbb{U}$, outer regularity is the only issue, i.e. we must show that all the critical values of φ lie in \mathbb{U} . Note first that ∞ is a critical point, with the corresponding critical value $\varphi(\infty) = 0 \in \mathbb{U}$. As for the critical points in the finite plane: these are the critical points of the denominator—the points z at which the derivative of the denominator vanishes; equivalently, the points z for which:

$$f_d(z) := 1 + 2z + 3z^2 + \dots + dz^{d-1} = 0. \quad (23)$$

It will be easier to deal with

$$g_d(z) := (1 - z)f_d(z) = 1 + z + z^2 + \dots + z^{d-1} - dz^d,$$

which adds to the zeros of f_d an additional one at $z = 1$. A simple exercise involving Rouché's theorem shows that g_d has, for each $R > 1$, d zeros in the disc $R\mathbb{U}$, hence d zeros in $\overline{\mathbb{U}}$, and therefore f_d has $d - 1$ zeros (counting multiplicity) in $\overline{\mathbb{U}}$. To complete the proof note that φ takes the closed disc into the open disc with the exception of the point 1. Since $f_d(1) \neq 0$ we see that all critical values of φ belong to \mathbb{U} . \square

It's not difficult to construct other examples of outer regular and strongly outer regular mappings; e.g., $\varphi(z) = z/(a - z^n)$ is a self-map of \mathbb{U} whenever $|a| \geq 2$ and is strongly outer regular whenever the integer n exceeds 1.

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